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ON CONTROL UNDER INCOMPLETE INFORMATION

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We examine the problem of control under incomplete information on the system's phase states realized. The initial problem is stated as an encounter problem in a suitable differential game. We use the concepts and notation of the techniques in [1, 2].

1. Statement of the problem. We consider the system described by the equation

$$dy / dt = A(t)\, y + f^*(t, u, v), \quad u \in P,\ v \in Q \tag{1.1}$$

Here y is the object's phase vector; u and v are the control vectors of the first and second players; P and Q are compacts; the matrix $A(t)$ and the vector $f^*(t, u, v)$ are continuous functions. The subject of the paper is the problem of the first player (the ally [1]) about the encounter $\{y[\vartheta]\}_m \in M$ at a fixed instant ϑ with a target set M specified in the m-dimensional space of the first m coordinates $\{y\}_m$ of vector y. The problem is reduced by a suitable linear transformation to the encounter problem $x[\vartheta] \in M$ at instant ϑ for an m-dimensional vector x with the same set M, where the time variation of $x[t$ is described by the equation

$$dx / dt = f(t, u, v) \tag{1.2}$$

The peculiarity of the problem is that at each current instant $t \in [t_0, \vartheta]$ the first player knows only the information region $G[t]$ containing the realization $x[t]$. Therefore, the problem is reduced to one of control of realizations $G[t]$, which ensures the inclusion $G[\vartheta] \subset M$. The target set M is assumed bounded, convex and closed. We assume that the admissible regions $G[t]$ are also bounded, convex and closed. We distinguish the following cases of the use of the information for constructing the regions $G[t]$.

1°. The second player (opponent) may come to know both the realizations $x[t]$ of the phase vector as well as the realizations $u[t]$ of the first player's control at the current instant t. The first player knows at the current instant t the realizations $G[t]$ of the information region, but does not receive direct information on the realizations $v[t]$ of the second player's control.

2°. The second player may come to know the realizations $x[t]$, but has no direct information on the realizations $u[t]$. The first player knows the realizations $G[t]$, but does not receive direct information on the realizations $v[t]$.

3°. The second player may come to know $x[t]$. The first player knows $G[t]$ and can use the values of $v[t]$ in the controls producing the operating force $u[t]$, but he cannot use the values of $v[\tau]$ $(\tau < t)$ for making the realization $G[t]$ of the information region more precise.

4°. The second player may come to know $x[t]$. The first player knows $G[t]$ and can use the values of $v[t]$ in the devices developing the control $u[t]$; he can also use the values of $v[\tau]$ $(\tau \leqslant t)$ for constructing the realization $G[t]$.

In all the cases the first player's control procedure in the small half-intervals $t_* \leqslant t < t^*$ is based on the following concept of his program $U[t_*, t^*)$, which is used here to construct analogs of the strategies in [1]. In each case we first define the concept of an elementary program $U_e[t_*, t^*)$, which we next extend to the more general concept of a program $U[t_*, t^*)$. An elementary program, specified for $t_* \leqslant t < t^*$, is a piecewise-constant function $u[t] \in P$ or a function piecewise-constant in t (e.g., a probability measure $\mu_t(du)$ on P) or a function $u(t, v) \in P$ piecewise-constant in t and Borel measurable in $v \in Q$ (respectively, in the cases 1° or 2°, or 3° and 4°); it is defined as an operation which forms the point sets

$$G_1[t_*, t^*] = \text{co}\left\{x : x = \int_{t_*}^{t^*} f(t, u[t], v(t))\, dt;\ v(t) \in Q\right\} \tag{1.3}$$

$$G_2[t_*, t^*] = \left\{x : x = \int_{t_*}^{t^*} \int_{P\times Q} f(t, u, v)\, \mu_t(du)\, \nu_t(dv);\ \nu_t(dv)\right\} \tag{1.4}$$

where $\nu_t(dv)$ ranges over the set of all possible probability measures $\nu_t(dv)$ on Q weakly measurable in t

$$G_3[t_*, t^*] = \text{co}\left\{x : x = \int_{t_*}^{t^*} f(t, u(t, v(t)), v(t))\, dt;\ v(t) \in Q\right\} \tag{1.5}$$

$$G_4[t_*, t^*] = x = \int_{t_*}^{t^*} f(t, u(t, v(t)), v(t))\, dt; v(t) \in Q \tag{1.6}$$

where the symbol $\text{co}\{x : x = \ldots\}$ denotes the closed convex hull of the corresponding set of vectors x; in equalities (1.3) and (1.5) $v(t)$ ranges over the set of all possible Lebesgue-measurable functions $v(t) \in Q$, while in equality (1.6) $v(t)$ it is some such fixed function.

The convex closed bounded region $G_j[t_*, t^*]$ is characterized by its support function

$$g_j(l; t^*, t_*, U) = \max_x l'x, \quad x \in G_j \tag{1.7}$$

(The vectors are treated as column-vectors, the prime denotes transposition, $\|l\|$ denotes the Euclidean norm vector l.) The functions $g_j(l; t^*, t_*, U)$ of the variable l are looked upon as elements of a Hilbert space (see [1]), H of functions $h(l)$ square-integrable on the sphere $\|l\| \leqslant 1$; the scalar product $\langle h\cdot g\rangle$ in H is defined by the equality

$$\langle h\cdot g\rangle = \int_{\|l\|\leqslant 1} h(l)\cdot g(l)\, d\{l\} \tag{1.8}$$

hence the norm $\|h\|_H$ is defined by the equality

$$\|h\|_H = \langle h\cdot h\rangle^{1/2} \tag{1.9}$$

For a fixed value of j we set up in the closed space H the convex hull $\text{co}\{g_j\}$ of the set of all support functions $g_j(l; t^*, t_*, U_e)$ corresponding to all possible elementary programs $U_e[t_*, t^*)$ for fixed t_* and t^*. The operation which forms the set $G[t_*,$

$t^*]$ corresponding to some support function $g(l;\ t^*,\ t_*, U) \in \text{co}\ \{g_j\}$ is now called the program $U\ [t_*,\ t^*)$ for the given value of j.

The fixed region $G\ [t_*]$ and program $U\ [t_*,\ t^*)$ define the attainability region $G\ (t^*,\ t_*,\ G\ [t_*],\ U)$ which comprises all points $x^* = x_* + x$, where $x_* \in G\ [t_*]$ and $x \in G\ [t_*,\ t^*]$. The time variation of regions $G\ [t]$ is subject to the following conditions (see [1]). Suppose that at the instant $t = t_*$ the region $G\ [t_*]$ has been realized and that for the half-interval $[t_*,\ t^*)$ the first player has chosen some program $U\ [t_*,\ \ t^*)$. Then, only a region $G\ [t^*]$ which satisfies the inclusion

$$G\ [t^*] \in G\ (t^*,\ t_*,\ G\ [t_*],\ U) \tag{1.10}$$

can be realized at the instant $t = t^*$. In addition, we assume that the region $G\ [t_*]$ and the program $U\ [t_*,\ t^*)$ delineate the family $\{G^*\ [t^*];\ G\ [t_*],\ U\ [t_*,\ t^*)\}$ of possible realization of regions $G^*\ [t^*]$ satisfying condition (1.10). Thus, only the region

$$G\ [t^*] \in \{G^*\ [t^*];\ G\ [t_*],\ U\ [t_*,\ t^*)\} \tag{1.11}$$

can be realized at the instant $t = t^*$. The properties of admissible families (1.11) will be stated later on.

The first player must form his own control on a discrete-time scheme based on a suitable partitioning $\Delta = \{\tau_i;\ \tau_0 = t_0,\ i = 1, \ldots,\ n,\ \tau_n = \vartheta\}$ of the t-axis into half-intervals $[\tau_i,\ \tau_{i+1})$. Each program $U\ [\tau_i,\ \tau_{i+1})$ chosen by the first player at the instant τ_i for the next half-interval $[\tau_i,\ \ \tau_{i+1})$ is determined by the realization $G\ [\tau_i]$ and, possibly, by the realizations at instant τ_i of some other auxiliary variables which the first player finds it convenient to form in his control devices. Let $M^{(\varepsilon)}$ be the Euclidean ε-neighborhood of the set M. Then, the first player's task is to construct a control procedure which is determined by the programs $\{U\ [\tau_i,\ \ \tau_{i+1});\ G\ [\tau_i], \ldots\}$ operating in the small intervals $[\tau_i,\ \tau_{i+1})$ and which ensures, for every preselected value of $\varepsilon > 0$, the inclusion

$$G\ [\vartheta] \subset M^{(\varepsilon)} \tag{1.12}$$

for all possible realizations of regions $G\ [t]$ satisfying conditions (1.10) and (1.11), if only the step $\delta\ (\Delta) = \max_i\ (\tau_{i+1} - \tau_i)$ of partitioning Δ is sufficiently small.

2. Formalization of the problem. The time variation of the information regions $G\ [t]$ is treated as a motion $g\ [l;\ t]$ in the functional phase space H containing, according to Sect. 1, the support functions

$$g\ [l;\ t] = \max_x\ l'x, \quad x \in G\ [t] \tag{2.1}$$

of regions $G\ [t]$. Let $g\ [l;\ \ t_*]$ be the support function for region $G\ [t_*]$. The support function $g\ (l;\ t^*,\ t_*,\ g\ [l;\ t_*],\ U)$ of the attainability region $G\ (t^*, t_*,\ G\ [t_*],\ U)$ is defined by the equality

$$g\ (l;\ t^*,\ t_*,\ g\ [l;\ t_*],\ U) = g\ [l;\ t_*] + g\ (l;\ t^*,\ t_*,\ U) \tag{2.2}$$

where the quantity $g\ (l;\ t^*,\ t_*,\ U)$ has been defined in Sect. 1 above.

As a special case of condition (1.11) we choose the following inequality (see [1]):

$$g\ [l;\ t] + g\ [-l,\ t] \leqslant \varphi\ (t,\ l) \tag{2.3}$$

where we assume the function $\varphi\ (t,\ l)$ to be nonincreasing and differentiable in t. In the general case we rewrite condition (1.11) for the support function $g\ [l,\ t]$ again as an inclusion

$$g\,[l;\ t^*] \in \{g^*[l;\ t^*];\ g\,[l;\ t_*],\ U\,[t_*,\ t^*)\} \tag{2.4}$$

We make the following assumptions concerning condition (2.4): (1) generally, condition (2.4) does not include the requirement that $g\,[l;\ t]$ be a support function for some region $G\,[t]$, it can be applied to some elements $g\,[l,\ t] \in H$ which are not support functions; (2) the problems appearing below on the minimum of (3.1) and (4.1) under condition (2.4) have the solutions p° and U^g needed in Sects. 3 and 4; (3) let the function $g\,[l;\ t^*]$ be formed in $g\,[l,\ t_*]$, $U\,[t_*,\ t^*)$ and $p\,(l)$ in accordance with (2.6) (see below) and let it satisfy inclusion (2.4) for $t = t^*$; then for the same program $U\,[t_*,\ t^*)$ we can find a segment $p_*\,(l) \leqslant p\,(l)$ such that the function $g_*\,[l;\ t^*] = g\,[l;\ t_*] + g\,(l;\ t^*,\ t_*,\ U) - p_*\,(l)$ again satisfies inclusion (2.4) for $t = t^*$ with the inequality

$$\|p_*(l)\|_H \leqslant \beta\,(t^* - t_*) \quad (\beta = \text{const}) \tag{2.5}$$

also satisfied. All the listed assumptions are satisfied in the special case (2.3) of condition (2.4). Thus, any function $g\,[l;\ t_*] \in H$, of the form

$$g\,[l;\ t^*] = g\,[l;\ t_*] + g\,(l;\ t^*,\ t_*,\ U) - p\,(l) \tag{2.6}$$

where the segment $p\,(l)$ is a nonnegative function from H, such that condition (2.4), or, in the special case, condition (2.3) (for $t = t^*$) , is satisfied, is called the motion $g\,[l;\ t^*]$ from the position $g\,[l;\ t^*] \in H$ generated by the program $U\,[t_*,\ t^*)$.

We seek the resolving control procedure in a scheme with a guide (see [1]). We describe the states of the guide model also by the elements $w\,[l;\ t] \in H$. We do not require that the function $w\,[l;\ t] \in H$ necessarily be a support function for some region $G\,[t]$. The guide's motion $w\,[l;\ t^*]$ generated by some program $U\,[t_*,\ t^*)$ from the position $w\,[l;\ t_*)$ is defined at first, by analogy with (2.6), by the formula

$$w^*\,[l;\ t^*] = w\,[l;\ t_*] + g\,(l;\ t^*,\ t_*,\ U) - p\,(l),\ p\,(l) \geqslant 0 \tag{2.7}$$

and then we allow a further change in it by a jump $w^*\,[l;\ t^*] \to w\,[l;\ t^*]$ satisfying the condition

$$\|\Delta w\|_H = \|w\,[l;\ t^*] - w^*\,[l;\ t^*]\|_H \leqslant \alpha\,(\delta)\,(\tau_{i+1} - \tau_i)$$

where $\alpha\,(\delta) \to 0$ as $\delta \to 0,\ \tau_{i+1} - \tau_i \leqslant \delta$.

Now, the collection

$$\begin{aligned}&\{U^g\,([\tau_i,\ \tau_{i+1});\ g\,[l;\ \tau_i],\ w\,[l;\ \tau_i])\\ &U^w([\tau_i,\ \tau_{i+1});\ g\,[l;\ \tau_i],\ w\,[l;\ \tau_i],\ g\,[l;\ \tau_{i+1}])\\ &p^w\,(l;\ g\,[l;\ \tau_i],\ w\,[l;\ \tau_i],\ g\,[l;\ \tau_{i+1}])\\ &w\,[l;\ \tau_{i+1};\ w\,[l;\ \tau_i],\ U^w,\ p^w,\ g[l;\tau_{i+1}]]\} \qquad (i = 0, 1, \ldots, \tau_0 = t_0)\end{aligned}$$

where the programs $U^g\,[\tau_i,\ \tau_{i+1})$ determine the system's motion $g\,[l;\ t]$ in accord with conditions (2.6) and (2.4) (for $t_* = \tau_i$, $t^* = \tau_{i+1}$ and $U = U^g$), while the programs $U^w\,[\tau_i,\ \tau_{i+1})$, the segment $p^w\,(l)$ and the function $w\,[l;\ \tau_{i+1}]$ determine successively the guide's motion $w\,[l;\ t]$ in the intervals $[\tau_i,\ \tau_{i+1}]$, is called the first player's strategy U (a pure strategy $U^{(1)}$ in Case 1°, a mixed strategy $U^{(2)}$ in Case 2°, a counterstrategy $U_3^{(V)}$ or a counterstrategy $U_4^{(V)}$ in Cases 3° or 4°). Thus, in the present formalization it remains for the second player to choose the segments $p\,(l) = p^g\,[l;\ \tau_i]$ $(i = 0,\ 1, \ldots)$ in the motion $g\,[l,\ t]$ in (2.6) of the given controlled system.

Let L be the set of elements $h\,(l)$ of H, which satisfy the condition $h\,(l) \leqslant m\,(l)$, where $m\,(l)$ is the support function of set M. We denote the ε-neighborhood of set L

in the metric of space H by $L^{(\varepsilon)}$. The introduced concepts lead to the following formalization of the original problem on encounter with set M extensively described in Sect. 1.

Problem 2.1. We consider the Cases $1°-3°$ or $4°$. The instant ϑ and the initial position $g[l;\ t_0]$ $(t_0 < \vartheta)$ are given. Find the strategy U which ensures, for every preselected value of $\varepsilon > 0$, the inclusion

$$g[l;\ \vartheta] \in L^{(\varepsilon)} \tag{2.8}$$

for every motion $g[l;\ t]$ generated by it from the given position $g[l;\ t_0]$, if only the step $\delta(\Delta) = \max_i (\tau_{i+1} - \tau_i)$ of the partitioning Δ $\{\tau_i\}$ of the t-axis is sufficiently small.

3. Stable bridge. As in a position differential game with complete information [1], in the considered case of incomplete information being the construction of the procedure for control with a guide is based on the concept of a stable bridge W in the space $\{t,\ H\}$, which must connect the initial position $g[l;\ t_0]$ with the target set L and along which by suitable action the guide's motion $w[l;\ t]$ can be led to set L at the instant $t = \vartheta$, while at the same time compelling the system's motion $g[l;\ t]$ and the guide's motion $w[l;\ t]$ to track each other.

Let $\lambda(l)$ be some nonnegative function from H and let $g[l;\ t_*]$ be some admissible element from H, i.e. a function $g[l;\ t_*] = h(l) \in H$ which does not contradict condition (2.4), although $g[l;\ t]$ is possibly but not necessarily the support function for some suitable region $G[t]$. Further, let $U[t_*,\ t^*)$ be some program. By the symbol $p^\circ(l;\ t^*,\ t_*,\ g[l;\ t_*],\ U,\ \lambda)$ we denote the extremal segment $p(l)$ in equality (2.6) which among all the segments $p(l)$ satisfying condition (2.4), also satisfies the following condition of minimum:

$$\langle p^\circ(l;\ t^*,\ t_*,\ g[l;\ t_*],\ U,\ \lambda)\cdot\lambda(l)\rangle = \min_{p(l)} \langle p(l)\cdot\lambda(l)\rangle \tag{3.1}$$

We assume (see Sect. 2) that condition (2.4) is such that the definition of the extremal segment p° as a function (perhaps nonunique) of the arguments $t_*,\ t^*,\ g[l;\ t_*],\ U$ and λ is well posed. Every function $p^*(l)$ defining the function

$$h^*(l) = g[l;\ t_*] + g(l;\ t^*,\ t_*,\ U) - p^*(l)$$

contained in the closed convex hull of the functions

$$h(l) = g[l;\ t_*] + g(l;\ t^*,\ t_*,\ U) - p^\circ(l;\ t^*,\ t_*,\ g[l;\ t_*],\ U,\ \lambda)$$

where the segments $p^\circ(l)$ have already been determined by the minimum condition (3.1), is also (formally) called an extremal segment $p^\circ(l,\ t^*,\ t_*,\ g[l;\ t_*],\ U^*,\ \lambda)$. Here the function $g[l;\ t_*]$ and $\lambda(l)$ are fixed, and U is an arbitrary program $U[t_*,\ t^*)$.

Now let W be some closed set in space $\{t,\ H\}$. The symbol $W(t)$ denotes the intersection of W with the hyperplane $t = \text{const}$. We say that set W forms a stable bridge reaching L at the instant ϑ if the following conditions are fulfilled:

1) for any $\varepsilon > 0$ if $w[l;\ \vartheta] \in W(\vartheta)$ and $w[l;\ \vartheta] \leqslant g[l;\ \vartheta]$, $\|w[l;\ \vartheta] - g[l;\ \vartheta]\|_H \leqslant \varepsilon$, where $g[l;\ \vartheta]$ is the support function for some possible realization of region $G[\vartheta]$, then $w[l;\ \vartheta] \in L^{(\varepsilon)}$;

2) for any sufficiently small $\alpha > 0$ we can find $\delta > 0$ satisfying the following

condition: suppose that we have chosen some values $t_* \in [t_0, \vartheta)$ and $t^* \in (t_*, \vartheta]$, $t^* - t_* \leqslant \delta$, an element $w[l;\ t_*] \in W(t_*)$ and a nonnegative function $\lambda(l) \in H$, satisfying the following condition: the function $g[l;\ t_*] = \lambda(l) + w[l;\ t_*]$ is the support function for some possible region $G[t_*]$, and, finally, suppose that we have chosen some support function $g[l;\ t^*]$ satisfying condition (2.4) for some choice of program U^*; then we can find at least one program $U[t_*, t^*)$ which ensures the inequality

$$\|w^*[l;\ t^*] - w[l;\ t^*]\|_H \leqslant \alpha(t^* - t_*)$$
$$w^*[l;\ t^*] = \min\{g[l;\ t^*];\ w[l;\ t_*] + g(l;\ t^*,\ t_*,\ U) - p^\circ(l;\ t^*,\ t_*,\ g[l,\ t_*],\ U,\ \lambda)\}$$
$$w[l;\ t^*] \in W(t^*), \quad w[l;\ t^*] \leqslant g[l;\ t^*] \qquad (3.2)$$

for at least one extremal segment $p^\circ(l;\ t^*,\ t_*,\ g[l;\ t_*],\ U,\ \lambda)$ corresponding to this program $U[t_*, t^*)$.

4. Extremal strategy. We now construct the extremal strategy U. Let W be some stable bridge reaching L at instant ϑ. The strategy

$$U^\circ = \{U^g,\ U^w,\ p^w,\ w(l;\ \tau_{i+1})\}$$

is said to be the strategy U° extremal to this bridge W. It is defined as follows: suppose that certain values

$$g[l;\ \tau_i],\ w[l;\ \tau_i] \in W(\tau_i),\ \lambda[l;\ \tau_i] = g[l;\ \tau_i] - w[l;\ \tau_i] \geqslant 0$$

have been realized at the instant $t = \tau_i$ $(i = 0, 1, \ldots)$. We choose the program

$$U^g([\tau_i,\ \tau_{i+1});\ g[l;\ \tau_i],\ w[l;\ \tau_i])$$

from the condition of minimum

$$\langle\lambda[l;\ \tau_i]\cdot[g(l;\ \tau_{i+1},\ \tau_i,\ U^g) - p^\circ(l;\ \tau_{i+1},\ \tau_i,\ g[l;\ \tau_i],\ U^g,\ \lambda[l;\ \tau_i])]\rangle =$$
$$\min_U \langle\lambda[l;\ \tau_i]\cdot[g(l;\ \tau_{i+1},\ \tau_i,\ U) - p^\circ(l;\ \tau_{i+1},\ \tau_i,\ g[l;\tau_i],\ U,\ \lambda[l;\tau_i])]\rangle$$

where the $p^\circ(l)$ are extremal segments fully defined by the minimum condition (3.1). The chosen program $U^g[\tau_i, \tau_{i+1})$, together with the segment $p[l;\ \tau_{i+1}]$ selected by the second player, determines the motion

$$g[l;\ \tau_{i+1}] = g[l;\ \tau_i] + g(l;\ \tau_{i+1},\ \tau_i,\ U^g) - p[l;\ \tau_{i+1}]$$

Now the program $U^w([\tau_i, \tau_{i+1}), g[l; \tau_i], w[l; \tau_i], g[l; \tau_{i+1}])$ is chosen on the basis of property (2) of the stable bridge W from the condition (see (3.2))

$$w[l;\ \tau_{i+1}] \in W(\tau_{i+1}),\ w[l;\ \tau_{i+1}] \leqslant g[l;\ \tau_{i+1}] \qquad (4.1)$$

where

$$\|w[l;\ \tau_{i+1}] - w^*[l;\ \tau_{i+1}]\|_H \leqslant \alpha(\delta)[\tau_{i+1} - \tau_i]$$
$$w^*[l;\ \tau_{i+1}] = \min\{g[l;\ \tau_{i+1}],\ w[l;\ \tau_i] + g(l;\ \tau_{i+1},\ \tau_i,\ U^w) - p^\circ(l;\ \tau_{i+1},\ \tau_i,\ g[l;\ \tau_i]$$
$$U^w,\ \lambda[l;\ \tau_i])\},\ \alpha(\delta) \to 0$$

as $\delta = \max_i(\tau_{i+1} - \tau_i) \to 0$. Therefore, the segment $p^w(l;\ \tau_{i+1},\ \tau_i,\ g[l;\ \tau_i],\ w[l;\ \tau_i],\ g[l;\ \tau_{i+1}])$, defining the model's motion

$$w^*[l;\ \tau_{i+1}] = w[l;\ \tau_i] + g(l;\ \tau_{i+1},\ \tau_i,\ U^w) - p^w(l;\ \tau_{i+1}, \qquad (4.2)$$

$$\tau_i,\ g\,[l;\ \tau_i],\ w\ [l;\ \tau_i],\ g\,[l;\ \tau_{i+1}])$$

is determined by the equality

$$p^w = p^\circ\ (l;\ \tau_{i+1},\ \tau_i,\ g\,[l;\ \tau_i],\ U^w,\ \lambda\ [l;\ \tau_i]] + p_*{}^\circ\ (l)$$

where $p_*{}^\circ\ (l)$ is a nonnegative function specified by conditions (4. 1) and (4. 2).

The extremal strategy U° constructed in this way, together with the segments $p\ [l;\ \tau_i]$ selected by the second player, determines the system's motion $g\ [l;\ \tau_i]$ and the guide's motion $w\,[l;\ \tau_i]$ for $i = 0,\ 1,\ldots$ from the position $\{g\ [l;\ t_0],\ w\ [l;\ t_0]\}$, where at the instant $t_0 = \tau_0$ we choose $w\ [l;\ \tau_0] = g\ [l;\ \tau_0]$. From the conditions of bridge W stability based on the selection of all the functions listed above, it follows that the strategy U° extremal to this bridge ensures the preservation of $w\ [l;\ \tau_i] \in W\ (\tau_i)$ for all values of τ_i $(i = 0,\ 1,\ldots,n)$, i.e. up to the instant $\tau_n = \vartheta$, provided that $w\ [l;\ \tau_0] \in W\ (\tau_0)$.

5. Basic results. The following statement is valid.

Theorem 5. 1. Let $g\ [l;\ \ t_0] \in W\ (t_0)$, where W is a stable bridge reaching L at instant ϑ. Then the strategy U° extremal to this bridge solves Problem 2. 1.

The proof of Theorem 5. 1 similar to that of similar theorems in [1] for games with complete information, is based on the estimate

$$\|\,g\,[l;\tau_{i+1}] - w^*\,[l;\tau_{i+1}]\,\|_H{}^2 \leqslant \|\,g\,[l;\tau_i] - w\,[l;\tau_i]\,\|_H{}^2 + o\,(\tau_{i+1} - \tau_i)$$

where $o\ (\delta)$ is an infinitesimal of higher order than δ. The proof of (5. 1) differs in details only from that of the analogous estimate in [1].

6. Approximate control scheme. Henceforth, for definiteness we take condition (2. 4) in the form of inequality (2. 3). We describe the approximate procedures corresponding to the programs $U^g\,[\tau_i,\ \tau_{i+1})$ selected by the first player's extremal pure, mixed, or counter-strategy U° in cases 1°—4°, respectively. Thus, suppose that an extremal strategy U° prescribes for the half-interval $[\tau_i,\ \tau_{i+1})$ some program $U^g\,[\tau_i,\ \tau_{i+1})$. We divide the half-interval $[\tau_i,\ \tau_{i+1})$ into some sufficiently large number of smaller half-intervals $[\tau_k{}^{(i)},\ \tau_{k+1}^{(i)})$ $(k = 0,\ 1,\ 2,\ \ldots\ ,\ k^{(i)},\ \tau_0{}^{(i)} = \tau_i,\ \tau_{k(i)}^{(i)} = \tau_{i+1})$ and approximate the program $U^g\,[\tau_i,\ \tau_{i+1})$ by a suitable elementary program. The approximating elementary program $U_e{}^g\ [\tau_i,\ \tau_{i+1})$ must ensure the appropriate proximity of the functions $g\ (l;\ \tau_{i+1},\ \tau_i,\ U^g)$ and $g\ (l;\ \tau_{i+1},\ \tau_i,\ U_e{}^g)$. A suitable approximation is always possible under the definition introduced for the program $U^g\ [\tau_i,\ \tau_{i+1})$.

Let the indicated elementary program $U_e{}^g\ [\tau_i,\ \tau_{i+1})$ be determined in Cases 1°, 3° and 4° as a piecewise-constant in time t function $u^{(i)}\ [t]$ or $u^{(i)}\ (t,\ v)$, respectively. Then when the control is effected by a real system, it is the control $u = u^{(i)}\ [t]$ (or $(u^{(i)}\ (t,\ v\ [t]))$ that is actually fed into it in the half-interval $[\tau_i,\ \tau_{i+1})$. And that control, for any preselected value of $\varepsilon > 0$, ensures the inclusion

$$x\,[\vartheta] \in M^{(\varepsilon)} \qquad (6.\ 1)$$

when the steps $\delta^{(i)} = \max_i\ (\tau_{i+1} - \tau_i)$ and $\delta^{(i)} = \max_k\ (\tau_{k+1}^{(i)} - \tau_k{}^{(i)})$ are chosen sufficiently small.

In Case 2° let the approximating elementary program $U_e{}^g\ [\tau_i,\ \tau_{i+1})$ be determined by the piecewise constant in time function $\mu_t{}^{(i)}\ (du)$. Then, when the control is effected by a real system, it is the piecewise constant control $u\ [t] = u_k{}^{(i)}$ that is actually fed into it in the half-intervals $[\tau_k{}^{(i)},\ \ \tau_{k+1}^{(i)})$, where $u_k{}^{(i)}$ is the result of a random trial at choosing a vector u with the probability distribution $\mu_k^{(i)}(du) = \mu_t^{(i)}(du)$ $(\tau_k^{(i)} \leqslant t < \tau_{k+1}^{(i)})$.

On the assumption of stochastic independence of the controls $u[t]$ and $v[t]$ in the small half-intervals $[\tau_k^{(i)}, \tau_{k+1}^{(i)})$ (as in [1]) this control, for any preselected values of $\varepsilon > 0$ and $p < 1$, ensures inclusion (6.1) with a probability not less than p when the steps δ and $\delta^{(i)}$ are chosen sufficiently small. Further, when the control method described is put into effect, the realizations $g[l, \tau_k^{(i)}]$ should be increased by the sufficiently small quantity $\alpha(\tau_k^{(i)} - \tau_{k-1}^{(i)}) \| l \|$.

7. Alternative. Thus, according to what we have presented above, for solving Problem 2.1 and for a practical realization of the control u leading the motion $x[t]$ at a specified instant $t = \vartheta$ into a preselected small neighborhood $M^{(\varepsilon)}$ of set M, it is sufficient to know how to construct a stable bridge W in the space $\{t, H\}$, reaching L at the instant ϑ. Therefore, as in a differential game with complete information [1] the question arises of the existence of a suitable bridge W when Problem 2.1 has a solution, and of effective methods for the construction of the required bridge W. An affirmative answer is given below to the first question. The effective construction of stable bridges in the case being considered here of a game with inclomplete information, as in the analogous cases of games with complete information [1], is possible either on the basis of the extremal aiming method (see [1]), or in the form of a priori stable bridges (see [1]). However, the modification of these methods for their application to the encounter problems in a differential position game with incomplete information falls outside the scope of this article and will be the subject of another paper.

The question of the existence of a stable bridge W in space $\{t, H\}$ is answered in connection with the following theorem on the alternative (see the analogous case in [1]), which we cite here without proof.

Theorem 7.1. For given M and ϑ one of the following two statements is valid for every initial state $g[l; t_0]$ $(t_0 < \vartheta)$: either Problem 2.1 has a solution or (otherwise) a value of $\varepsilon > 0$ exists such that for any choice of partitioning $\Delta = \{\tau_i\}$ with a sufficiently small step and of control $u[t]$ successively chosen by the first player in the half-intervals $[\tau_i, \tau_{i+1})$, the second player can dispose of the segments $p[l; \tau_i]$ in the motions $g[l; \tau_i]$ given by (2.6) so as to exclude the inclusion $G[\vartheta] \subset M^{(\varepsilon)}$.

This theorem on the alternative, as for a game with complete information (see [1]) is proved as follows. From the region $t_0 \leqslant t \leqslant \vartheta$ in space $\{t, H\}$ we remove all those points $\{t, h(l)\}$, for each of which, as well as for the initial state $g[l, t] = h(l)$, the second statement of Theorem 7.1 is satisfied for at least one value of $\varepsilon > 0$; we also remove all the points $\{t, h(l)\}$ for which condition (2.3) is not satisfied. We are left with a point set W forming a stable bridge reaching M at instant ϑ. The stability Conditions (1) and (2) for W (see Sect. 3) are strongly satisfied. Namely: (1°) if $w[l; \vartheta] \in W(\vartheta)$, then $w[l, \vartheta] \in L$; (2°) suppose that we have chosen some values $t_* \in [t_0, \vartheta)$ and $t^* \in (t_*, \vartheta]$, an element $w[l; t_*] \in W(t_*)$ and a nonnegative function $\lambda(l) \in H$, then we can find at least one program $U[t_*, t^*)$ which ensures the inclusion $w[l; t^*] \in W(t^*)$ for the motion

$$w[l; t^*] = w[l; t_*] + g(l; t^*, t_*, U) - p^\circ(l; t^*, t_*, w[l; t_*] + \lambda(l), U[t_*, t^*), \lambda(l))$$

From such a construction of the stable bridge W follows the validity of the assertion that whenever Problem 2.1 is solvable for a given initial state $g[l; t_0]$, a stable bridge W exists, that reaches M at instant ϑ ; consequently, if only a solution of Problem 2.1

exists, it can always be constructed as a strategy U° extremal to a suitable stable bridge W. Such solution can be realized as the approximate procedure described in Sect. 6.

In conclusion we note that when all possible points $x^*[t^*]$ in the attainability region $G(t^*, t_*, G[t_*] = x[t_*], U[t_*, t^*])$, and only they, are the elements $G^*[t^*]$ of the family $\{G^*[t^*]\}$ of (1.11), we obtain the idea of a differential position game with complete information, in the formalization given in [1]. In a game with complete phase information the distinction between Cases 2° and 4° disappears and both these cases reduce to one and the same minimax differential game (see [1]). We take this opportunity to note that in the most general minimax differential position game the results for the nonlinear system $dx / dt = f(t, x, u, v)$ remain entirely unchanged if in the constructions of that game the realization $v[t]$ of the opponent's control at the current instant t is replaced by the quantity

$$v^-(t) = \lim \left[\int^t v[\xi] \frac{d\xi}{(t-\tau)} \right], \quad \tau \to t - 0 \tag{7.1}$$

for almost all values of t for which the limit (7.1) exists and, it can be assumed that $v^-(t)$ is an arbitrary quantity $v \in Q$ for those values of t for which the limit (7.1) does not exist. According to results in the theory of functions of a real variable, for every choice of a measurable realization $v[t]$ the equality $v[t] = v^-(t)$ is valid for almost all values of $t \in [t_0, \vartheta]$. Hence, the Euler base broken lines $x_\Delta[t]$ which satisfy the equations

$$dx_\Delta/dt = f(t, x_\Delta[t], u(\tau_i, x_\Delta[\tau_i], v[t]), v[t])$$
$$(\tau_i \leqslant t < \tau_{i+1})$$

(see [1]), are absolutely unaltered when $v[t]$ is replaced by $v^-(t)$ and consequently, the motions $x[t]$, which are the limits of these Euler broken lines, are entirely unchanged. Thus, in the minimax differential position game in the considered formalizations we do not require, strictly speaking, at each current instant t information on the current realization $v[t]$ of the opponent's control at that same instant; it suffices to know only the history $\{v[\tau]\}$ $(\tau < t)$ of this realization up to the instant t. Moreover, as a consequence of the limitation of the choice available to the opponent by the measurable realizations $v[t]$ and of the equality $v[t] = v^-(t)$ valid for such realizations, for almost all values of t, this assertion may have a formal sense.

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